# Bayesian maximum entropy solution of the stochastic axisymmetric displacements problem around an excavation, in light of site-specific information

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ABSTRACT

This work presents a Bayesian Maximum Entropy (BME) approach to solve the Stochastic Differential Equation (SDE) representing the axisymmetric radial displacements calculation problem for a circular excavation in an elastoplastic rock subjected to an initial hydrostatic stress field. The representation of the problem by an SDE instead of an ordinary differential equation has an important advantage: In addition to the physical law, the BME solution can assimilate other sources of general and site- specific information such as measurements in the case of this work. Also, the final solution of the stress- strain law is given in the form of a probability distribution of possible displacement values at each point in the direction of the radius of the excavation. This is the most complete way to describe a stochastic solution and provides considerable flexibility in selecting the displacements distribution that is more representative of the physical situation.

## 1. INTRODUCTION

Modeling techniques generating predictive distributions of critical parameters across space and time in underground excavations are common in earth related sciences (e.g. Yamamoto et al. 2000, Webster 2003). In view of the uncertainty characterizing these distributions, the physical equation has often the form of a Stochastic Differential Equation (SDE). In this case, the maps produced by the model represent the solution of the SDE given a set of possibly uncertain boundary/ initial conditions of the situation. In general, two groups of techniques are commonly used for the solution of a SDE: (1) one group focuses on obtaining solutions that are valid for specific realizations of the SDE coefficients (e.g., Monte Carlo or realization based techniques (Adler, 1992)), and (2) another group is concerned with the estimation of stochastic moments (Kitanidis, 1986). Both groups have advantages and drawbacks, and they should be viewed as complementary tools for determining the behavior of the stochastic solutions (Jordan and Smith, 1987).

A different conceptual framework for obtaining stochastic solutions of physical SDE is suggested by the spatiotemporal Bayesian maximum entropy (BME) mapping approach introduced by Christakos (1990, 1991). The implementation of the BME approach to solve a Physical SDE differs from most standard SDE techniques by distinguishing between three stages of physical knowledge processing as follows:

1. At the structural (prior) stage, BME generates an initial probability distribution across space and time based on the physical SDE as well as other forms of general knowledge (primitive equations, multiple- point statistics, etc.), whenever available.
2. At the metaprior stage, databases expressing site- specific states of knowledge (e.g., uncertain observations or frequency distributions) are transformed into an operator form suitable for further processing
3. At the integration (posterior) stage, the initial solution of (1) is enriched by

assimilating the site- specific data from (2). The final solution is not limited to a single realization but includes the complete probability law at each space/ time point.

As proposed by Christakos (1992, 2000), in theory two main techniques can be used in stage 1 above: The so-called A technique, which does not need to solve the stochastic moment equations associated with the physical law, whereas the so-called B technique requires the solution of the moment equations. Christakos and Christopoulos (1998) examined several theoretical features of these techniques. Serre and Christakos (1999) used a numerical approach based on the B technique to study Darcy's law representing groundwater flow in porous media, whereas Kolovos et al. (2002) developed a systematic computational approach based on A technique to solve a stochastic partial differential equation representing the advection- reaction distribution of a pollutant in a river.

In this work we apply the B technique in rock mechanics. Since no previous application of the BME methodology is reported on the field, we study the relatively simple case of solving the SDE representing the axisymmetric radial displacements around a circular excavation in an elastoplastic rock, subjected to an initial hydrostatic stress field. The data used in the study come from measurements across the mount Kallidromon tunnel under development by the National Railway Organization in central Greece.

## 2. ASSIMILATION OF GENERAL KNOWLEDGE BY THE BME APPROACH

### *2.1 Displacement Law as General Knowledge*

In the application presented in this work, the general knowledge involves a physical law that governs the stress- strain distribution within the yield zone formed around a circular excavation in massive, elastic rock subjected to an initial hydrostatic stress field as shown in Fig. 1. The radial displacement u within the yield zone is given by the law represented as a differential equation (Brady and Brown, 1994). The complexity of the underground spatial rock structure often makes it difficult to account deterministically for all the contributing parameters in the physical law. In this sense, it is preferable to seek for a stochastic solution of the displacement problem. More specifically, we consider the radial displacement around a circular excavation in an elastoplastic rock, in which case the physical differential equation is as follows:

$$\frac{d\mathbf{u}}{d\mathrm{r}} = -f\,\frac{\mathbf{u}}{\mathrm{r}} + (1-f)\frac{(p-p_1)}{2\mathbf{G}} \qquad (1)$$

where $\mathbf{u}$(r) denotes the radial displacement (in mm) inside the yield zone and it is positive in the direction outwards from the center of the tunnel, r is the distance from center (in m), $p$ is the hydrostatic pressure acting on the rock mass (in kPa), $p_1$ is the pressure on the boundary between elastic and fractured domains, $\mathbf{G}$ is the shear modulus (in kPa) and $f$ is an experimentally determined dilation constant. In order to account for random influences, equation (1) is considered stochastic, in which case $\mathbf{u}$(r) is modeled as a spatial random function of displacements (note that in (1), random variables are represented by bold

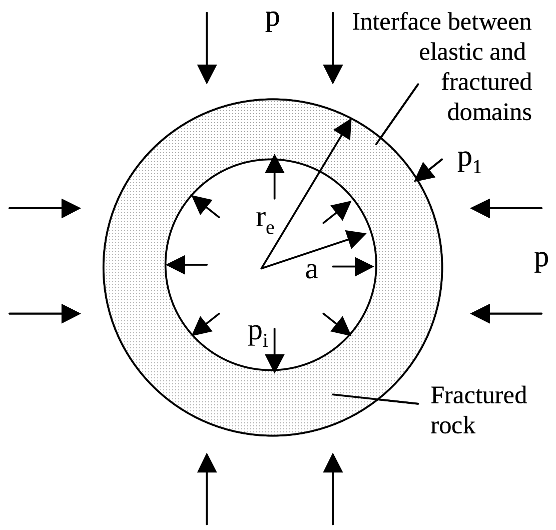


Figure 1: Stress distribution around a circular opening in a hydrostatic stress field.

letters). For (1) to be consistent, at least one of the "known" parameters must also be a random variable, so G is considered as a Gaussian random variable with mean $m_G$ and variance $\sigma_G^2$. Such a choice is justified on the basis of physical experience.

*2.2 General Knowledge-Based SDE Solution*

The following analytical solution to the differential equation (1) is considered in terms of stress field realizations:

$$\mathbf{u} = C\mathrm{r}^{-f} + \frac{(1-f)(p-p_1)}{2\mathbf{G}(1+f)}\mathrm{r} \tag{2}$$

where C is a constant of integration which may be evaluated by substituting the value of **u** at r = $r_e$, where $r_e$ is the radius of the interface between elastic and fractured domains (Fig. 1). This leads to the solution:

$$\frac{\mathbf{u}}{\mathrm{r}} = -\frac{(p-p_1)}{\mathbf{G}(1+f)}\left[\frac{(f-1)}{2} + \left(\frac{\mathbf{r_e}}{\mathrm{r}}\right)^{1+f}\right] \tag{3}$$

The boundary condition r = $r_e$ is also a Gaussian random variable with mean $m_{re}$ and variance $\sigma_{re}^2$. This variable is considered independent from **G**. Thus, the two random variables $\mathbf{r_e}$ and **G** generate the whole random function **u**(r) as shown schematically in Fig. 2.

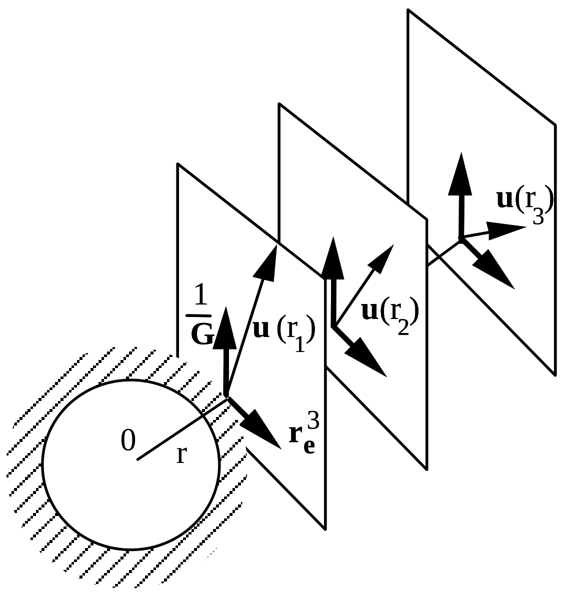


Figure 2: The two initial/ boundary random variables generate the random function **u**(r)

Substituting into (3) the known dilation parameter f = 2 and

$$a(r) = \frac{p-p_1}{6}\mathrm{r}, \quad b(r) = \frac{p-p_1}{3\mathrm{r}^2}$$

$$\mathbf{L} \equiv \frac{1}{\mathbf{G}}, \quad \mathbf{E} \equiv \mathbf{r_e}^3 \tag{4}$$

the solution becomes:

$$\mathbf{u}(\mathrm{r}) = a(\mathrm{r})\mathbf{L} + b(\mathrm{r})\mathbf{L}\cdot\mathbf{E} \tag{5}$$

By (4), the mean and variance of **L** and **G** can be approximated by (Papoulis, 1991):

$$m_L = \frac{1}{m_G} + \frac{\sigma_G^2}{m_G^3}, \quad \sigma_L^2 = \frac{1}{m_G^4}\sigma_G^2 \tag{6}$$

$$m_E = m_{re}^3 + 3m_{re}\sigma_{re}^2, \quad \sigma_E^2 = 3m_{re}^2\sigma_{re}^2 \tag{7}$$

From (5), (6) and (7) we can calculate the mean and variance of the random function **u**(r) at any point r inside the yield zone and also the centered covariance between two points $r_i$ and $r_j$ as follows (Appendix A):

$$\bar{\mathbf{u}}(\mathrm{r}) = a(r)m_L + b(r)m_L m_E \tag{8}$$

$$\begin{aligned}\sigma_{(r)}^2 &= a(r)^2\sigma_L^2 + 2a(r)b(r)\sigma_L^2 m_E + \\ &\quad b(r)^2\left[\sigma_L^2\left(m_E^2+\sigma_E^2\right)+\sigma_E^2 m_L^2\right]\end{aligned} \tag{9}$$

$$\begin{aligned}c_{(r_i,r_j)} &= a(r_i)a(r_j)\sigma_L^2 + \\ &\quad \left[a(r_i)b(r_j)+a(r_j)b(r_i)\right]\sigma_L^2 m_E + \\ &\quad b(r_i)b(r_j)\left[\sigma_L^2\left(m_E^2+\sigma_E^2\right)+\sigma_E^2 m_L^2\right]\end{aligned} \tag{10}$$

## 3. ASSIMILATION OF SITE-SPECIFIC KNOWLEDGE

Following the general knowledge assimilation stage is the metaprior stage, at which the site-specific knowledge is gathered and evaluated. In the case of this study, this knowledge consists of in situ measurements of radial displacements around the tunnel, and other experimentally determined parameters shown in Table 1.

Given the relations (8), (9) and (10), the joint Probability Density Function (PDF) $f_g(u_i,u_j)$ of displacements between two points $r_i$ and $r_j$ can be calculated (Appendix A). At the final (posterior or integration) stage, the general and the site-specific knowledge are combined and logically processed in order to yield the final

Table 1: Parameters and the values used in the study

| Parameter | Symbol | Value |
|---|---|---|
| Mean of Shear Modulus | $m_G$ | 25000kPa |
| Standard Deviation of Shear Modulus | $\sigma_G$ | 1000kPa |
| Mean of radius $r_e$ | $m_{re}$ | 7500mm |
| Standard Deviation of radius $r_e$ | $\sigma_{re}$ | 500mm |
| Experimental dilation parameter $f$ | $f$ | 2 |
| Hydrostatic pressure | $p$ | 2926kPa |
| Pressure at $r_e$ | $p_I$ | 1721kPa |

specific PDF $f_s(u_i/u_j=u)$. (Appendix B).
There are two available displacement

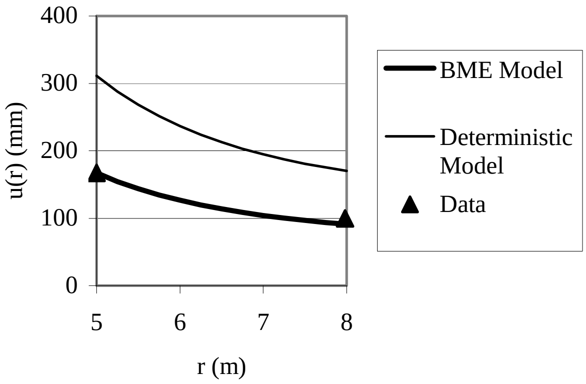


Figure 3: Mean values of the estimated displacements inside the yield zone

measurements; one on the tunnel wall and 5m from its center that gives -175 mm and one at 8 m from center that gives -93 mm. The first result is used as a datum, while the second is employed as a test point. An estimation value is calculated on each of a series of 15 equidistant points set across the radius, from 5 to 8 m from the center of the tunnel cross-section, inside the yield zone. At each estimation point $r_2$ we form the conditional PDF of the displacement at $r_2$, given the datum at r=5 m. Since this distribution is Gaussian, the most probable value coincides with its mean, which is denoted by (Appendix B):

$$\mathrm{E}\{\mathbf{u}(r_2)/\mathbf{u}(r_1)=u\}=\bar{\mathbf{u}}(r_2)+\frac{c(r_1,r_2)}{\sigma_1^{\ 2}}\left(u-\bar{\mathbf{u}}(r_1)\right) \quad (11)$$

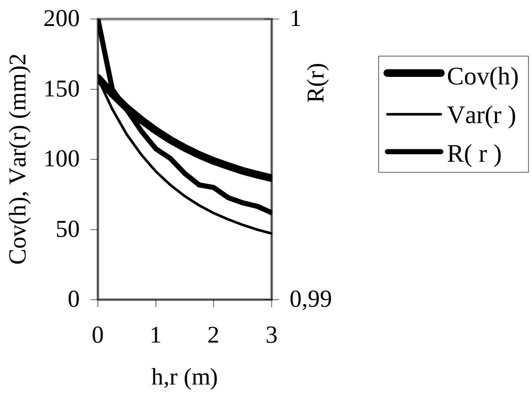


Figure 4: Variance, covariance and correlation coefficient R, relative to the tunnel wall, inside the yield zone

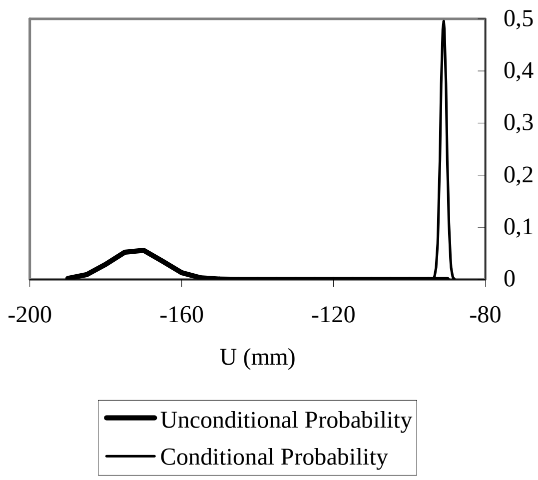


Figure 5: Displacement probabilities at the test point r=8 m before and after taking into account the datum at r=5 m

The above conditional mean at each point r is used as the estimation of the displacement u(r). The numerical results are plotted in Fig. 3

It is obvious that the fitting of the model is very satisfactory. Figure 5 shows that even if the unconditional PDF at the test point is wide, the accuracy is greatly improved by taking into account the sample value at the tunnel wall.

## 4. CONCLUSIONS

We have presented a novel method to solve the SDE representing the axisymmetric problem of calculation of radial displacements around a circular excavation in an elastoplastic rock, subjected to an initial hydrostatic stress field. The method is based on the BME theory and it introduces solutions at every point in the form of complete PDF, which do not only account for the physical law of interest but also for other site-specific sources of knowledge, such as data from in-situ measurements.

In the specific application, the adoption of BME methodology led to a significant improvement of accuracy, as compared to the standard deterministic estimation.Following the general knowledge assimilation

## APPENDIX A

Given the first and second order moments of two random variables, the bivariate PDF that describes their joint probability density is

$$f_g(u(r_i),u(r_j)) = \frac{1}{2\pi|c_u|^{1/2}}\exp\left[-\frac{1}{2}(\mathbf{u}-\overline{\mathbf{u}})^T c_u^{-1}(\mathbf{u}-\overline{\mathbf{u}})\right], \quad \text{(A1)}$$

$$where \quad (\mathbf{u}-\overline{\mathbf{u}})^T = (u(r_i)-\overline{\mathbf{u}}(r_i) \quad u(r_j)-\overline{\mathbf{u}}(r_j)), \quad and \quad c_u = \begin{pmatrix} c_{ii} & c_{ij} \\ c_{ji} & c_{ji} \end{pmatrix}$$

In order to calculate the means, variances and covariances in (A1) we have:

$$\overline{\mathbf{u}}(\mathrm{r}) = \overline{a(r)L + b(r)L\cdot E} = a(r)m_L + b(r)m_L m_E \quad \text{(A2)}$$

$$\sigma_{(r)}{}^2 = \overline{\mathbf{u}(r)^2} - \overline{\mathbf{u}}(r)^2 = \overline{(a(r)\mathbf{L}+b(r)\mathbf{L}\cdot\mathbf{E})^2} - \left(\overline{a(r)\mathbf{L}+b(r)\mathbf{L}\cdot\mathbf{E}}\right)^2 =$$
$$\overline{a(r)^2\mathbf{L}^2 + b(r)^2\mathbf{L}^2\mathbf{E}^2 + 2a(r)b(r)\mathbf{L}^2\mathbf{E}} - \left(a(r)^2 m_L{}^2 + b(r)^2 m_L{}^2 m_E{}^2\right) - 2a(r)b(r)m_L{}^2 m_E =$$
$$a(r)^2\sigma_L{}^2 + 2a(r)b(r)\sigma_L{}^2 m_E + b(r)^2\left[\sigma_L{}^2\left(m_E{}^2+\sigma_E{}^2\right)+\sigma_E{}^2 m_L{}^2\right] \quad \text{(A3)}$$

$$c_{(r_i,r_j)} = \overline{\mathbf{u}(r_i)\mathbf{u}(r_j)} - \overline{\mathbf{u}}(r_i)\overline{\mathbf{u}}(r_j) =$$
$$\overline{(a(r_i)\mathbf{L}+b(r_i)\mathbf{L}\cdot\mathbf{E})(a(r_j)\mathbf{L}+b(r_j)\mathbf{L}\cdot\mathbf{E})} - \left(a(r_i)\overline{\mathbf{L}}+b(r_i)\overline{\mathbf{L}}\cdot\overline{\mathbf{E}}\right)\left(a(r_j)\overline{\mathbf{L}}+b(r_j)\overline{\mathbf{L}}\cdot\overline{\mathbf{E}}\right) =$$
$$a(r_i)a(r_j)\sigma_L{}^2 + \left[a(r_i)b(r_j)+a(r_j)b(r_i)\right]\sigma_L{}^2 m_E + b(r_i)b(r_j)\left[\sigma_L{}^2\left(m_E{}^2+\sigma_E{}^2\right)+\sigma_E{}^2 m_L{}^2\right] \quad \text{(A4)}$$

## APPENDIX B

If the random variables $\mathbf{u}_1$ and $\mathbf{u}_2$ are jointly normal with zero mean, then (A1) rewrites as:

$$f(u(r_1),u(r_2)) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-R^2}}\exp\left[-\frac{1}{2(1-R^2)}\left(\frac{u_1{}^2}{\sigma_1{}^2} - \frac{2R\cdot u_1 u_2}{\sigma_1\sigma_2} + \frac{u_2{}^2}{\sigma_2{}^2}\right)\right] \quad \text{(B1)}$$

where $R$ the correlation coefficient of $\mathbf{u}_1$ and $\mathbf{u}_2$. The exponential above equals

$$\frac{\left(u_2 - \frac{R\sigma_2 u_1}{\sigma_1}\right)^2}{2\sigma_2{}^2(1-R^2)} - \frac{u_1{}^2}{2\sigma_1{}^2}$$

Division by $f(\mathrm{u}(\mathrm{r}_1))$ removes the second term and (B1) results:

$$f(u(r_1)/u(r_2)=u)=$$

$$\frac{1}{\sigma_2\sqrt{2\pi(1-R^2)}}\exp\left[-\frac{\left(u_2-\frac{R\sigma_2 u_1}{\sigma_1}\right)^2}{2\sigma_2^{\ 2}\left(1-R^2\right)}\right] \qquad \text{(B2)}$$

Now, for $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2 \neq 0$, replacing in (B2) $u_1$ and $u_2$ by $u_1$- $\bar{\mathbf{u}}_1$ and $u_2$- $\bar{\mathbf{u}}_2$, it is easily seen that

$$\mathrm{E}\{\mathbf{u}(r_2)/\mathbf{u}(r_1)=u\}=\bar{\mathbf{u}}(r_2)+\frac{c(r_1,r_2)}{\sigma_1^{\ 2}}\left(u-\bar{\mathbf{u}}(r_1)\right) \quad \text{(B3)}$$

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